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Finite quasi-quantum groups over abelian groups

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Abstract. This is a survey of some recent results on finite quasi-quantum groups over abelian groups. The works rely heavily on a unified and explicit formula of 3-cocycles on finite abelian groups. The formula reveals a useful separation of the 3-cocycles into two types, namely the abelian and the nonabelian ones. We give a brief summary to the classification of finite quasi-quantum groups of diagonal type induced by abelian 3-cocycles, and mention some initial advances of finite quasi-quantum groups of nondiagonal type induced by nonabelian 3-cocycles. Some problems for further research are also proposed.

1. Introduction

The classification problem of finite quasi-quantum groups (i.e., quasi-Hopf and coquasi-Hopf algebras) is motivated mainly by the theory of finite tensor categories [**17**] initiated by Etingof and Ostrik in the beginning of the century. As is clear that the general classification problem is far out of reach, it is necessary to narrow the scope and focus first on some interesting classes of tensor categories and quasi-quantum groups. In their pioneering work [**13**], Etingof and Gelaki proposed to classify nonsemisimple pointed finite tensor categories. By pointed it is meant that the simple objects are invertible. There are multifold reasons for this restriction: first, this kind of reduction is standard and powerful in representation theory; second, this class of tensor categories are essentially concrete, i.e., they admit quasi-fiber functors and they can be realized as the module categories of finite-dimensional elementary quasi-Hopf algebras by the Tannakian formalism [**17**]; third, this theory is a natural generalization of the deep and beautiful theory of finite-dimensional pointed Hopf algebras, see for example [**1**–**3**, **9**, **19**, **22**] among many other works.

Similar to the Hopf situation, the familar reduction procedures of degeneration and deformation (or lifting), factorization and bosonization (or biproduct) are useful in the quasi-Hopf situation as well. Let H be a finite-dimensional pointed coquasi-Hopf algebra with associator Φ. By $\{H_n\}_{n\geq 0}$ we denote its coradical filtration, and

$$\operatorname{gr} H = H_0 \oplus H_1/H_0 \oplus H_2/H_1 \oplus \cdots$$

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the corresponding coradically graded coalgebra. Then naturally $\operatorname{gr} H$ inherits from H a graded coquasi-Hopf algebra structure and is called a degeneration of H. The corresponding graded associator $\operatorname{gr}\Phi$ satisfies $\operatorname{gr}\Phi(\bar{a},\bar{b},\bar{c})=0$ for all homogeneous $\bar{a},\bar{b},\bar{c}\in\operatorname{gr} H$ unless they all lie in H_0. In particular, H_0 is a coquasi-Hopf subalgebra and it turns out to be $(kG,\operatorname{gr}\Phi)$ for $G=G(H)$, the set of group-like elements of H. We call a pointed coquasi-Hopf algebra H coradically graded if $H\cong\operatorname{gr} H$ as coquasi-Hopf algebras. Thus, any finite-dimensional pointed coquasi-Hopf algebra has a coradically graded degeneration. Conversely, one may lift or deform coradically graded pointed coquasi-Hopf algebras to get general ones. Now let $H=\oplus_{i\geq 0}H_i$ be a coradically graded pointed coquasi-Hopf algebra. Then H can be factorized as the form $R\#\Bbbk G$ where R is a connected graded Hopf algebra in a twisted Yetter-Drinfeld category of G. This reduces the study of coradically graded pointed coquasi-Hopf algebras to that of connected graded Hopf algebras in twisted Yetter-Drinfeld categories of groups. One can recover the former by the latter via the bosonization procedure.

To the best of our knowledge, so far most of the classification results on finite-dimensional quasi-Hopf algebras are in the coradically graded case. In [**13**, **14**], Etingof and Gelaki obtained a series of classification results about graded elementary quasi-Hopf algebras over cyclic groups of prime order; in [**15**, **18**], they studied graded elementary quasi-Hopf algebras over general cyclic groups and their liftings. One main achievement of this series of works is a complete classification of elementary quasi-Hopf algebras of rank 1. More importantly, a novel method of constructing genuine quasi-Hopf algebras (i.e., not twist equivalent to ordinary Hopf algebras) from known pointed Hopf algebras is invented. Along the same vein, Angiono classified in [**5**] finite-dimensional elementary quasi-Hopf algebras over cyclic groups whose orders have no small prime divisors. The basic idea of Etingof and Gelaki in [**13**–**15**] is embedding a genuine elementary quasi-Hopf algebras into an elementary quasi-Hopf algebra, possibly up to twist equivalence. The crux of these constructions is that there is a resolution for any given 3-cocycle on a cyclic group, namely, for any 3-cocycle σ on $\mathbb{Z}_n=\langle g|g^n=1\rangle$, the pull-back $\pi^*(\sigma)$ along the natural projection $\pi:\mathbb{Z}_{n^2}\to\mathbb{Z}_n$ is a 3-coboundary on $\mathbb{Z}_{n^2}$.

The previous idea can be easily extended to a full generality theoretically. However, there are several serious problems on the way of an explicit fulfillment. Some results in this direction are included in [**27**, **28**, **33**].

(1) The first problem is to find a resolution, if possible, for any normalized 3-cocycle on a finite group. This essentially lies in the cohomology theory of finite groups and is naturally split into two separated, namely the abelian and the nonabelian, cases. It is not surprised that the abelian case is relatively easier to handle. By extending the idea of [**30**], we are able to give a unified and explicit formula for a complete set of representatives of normalized 3-cocycles on any finite abelian groups. These 3-cocycles are further split into two types, that is the abelian and the nonabelian cocycles, according to whether or not the simple objects of the associated twisted Yetter-Drinfeld categories are all 1-dimensional. Moreover, we show that a 3-cocycle is resolvable by a finite abelian group if and only if it is abelian and we give an explicit resolution if this is indeed the case. For nonabelian 3-cocycles, there is no chance to resolve them and the associated twisted Yetter-Drinfeld categories are much more complicated.

For nonabelian groups, even the formulas for normalized 3-cocycles are not available yet.

(2) The second problem is to give a clear description of the Nichols algebras in a twisted Yetter-Drinfeld category ${}_{\mathbb{G}}^{\mathbb{G}}\mathcal{YD}^{\Phi}$. In case $\mathbb{G}$ and Φ are abelian, we can transform them to those in the usual Yetter-Drinfeld category ${}_{G}^{G}\mathcal{YD}$ by a delicate manipulation, where G is a finite abelian group with canonical projection $\pi : G \to \mathbb{G}$ such that $\pi^*(\Phi)$ is a 3-coboundary on G. The possibility of such a transformation is guaranteed since all the abelian 3-cocycles are resolvable. Then by combining Heckenberger's classification of arithmetic root systems [**22**], we achieve a complete classification of diagonal Nichols algebras with arithmetic root systems in ${}_{\mathbb{G}}^{\mathbb{G}}\mathcal{YD}^{\Phi}$. When the 3-cocycle Φ is nonabelian, the associated Nichols algebras become very complicated. Though we know in most cases they are infinite-dimensional, for the moment it is still far from getting a complete classification of the finite-dimensional ones.

(3) The third is the generation problem. In the diagonal case, with the transformation in (2) we can also reduce the problem of generation into that of Nichols algebras in the usual Yetter-Drinfeld categories of finite abelian groups. With a help of Angiono's result [**6**], we prove that finite-dimensional pointed coquasi-Hopf algebras of diagonal type are generated by group-likes and skew-primitive elements. Then we obtain a complete classification of finite-dimensional coradically graded pointed coquasi-Hopf algebras of diagonal type in a conceptual way. In nondiagonal case, a satisfactory theory of Nichols algebras is not available yet. So far only those Nichols algebras of semisimple twisted Yetter-Drinfeld modules with few summands are touched. It turns out that if the number of summands is less than or equal to 2, then we are able to make a connection from this to the diagonal case.

(4) The fourth problem is find a method to turn the conceptual classification into an operable procedure of constructions. In the diagonal case, for any given finite abelian group with fixed 3-cocycle and a compatible arithmetic root system, the construction is essentially a computational problem of linear congruence equations. We find two efficient ways, for most cases, to generate series of new genuine finite-dimensional pointed coquasi-Hopf algebras. In the nondiagonal case, basically we know nothing except very few sporadic examples.

There are also some other approaches to finite quasi-quantum groups and pointed finite tensor categories. For example, in [**7**, **8**] Angiono, Galindo and Pereira studied the de-equivariantization of the category of comodules over a Hopf algebra and characterized pointed finite tensor categories over abelian groups constructed as de-equivariantizations of tensor categories of comodules over finite-dimensional pointed Hopf algebras. Some results obtained in *ibid.* are close to ours in [**27**, **28**]. Specifically, abelian cocycles are called trivializable in the paper [**7**] by Angiono-Galindo, where the same description on the commutativity of the twisted Drinfeld double, and also an explicit characterization using results by Breen on cohomology [**11**] are found. In the same work the result about generation in degree one is included as well as a classification result in terms of de-equivariantizations when the 3-cocycle is abelian. In our earlier works [**25**, **26**], we proposed an approach via quivers and representations. In particular, a Drozd trichotomy of graded pointed finite tensor categories was obtained in [**29**, **31**] by the well known theory of representation types of algebras. Due to the lack of space, we do not include any more details of these aspects.

The rest of the paper is organized as follows. In Section 2, we recall the formulas of normalized 3-cocycles on finite abelian groups and the application to a classification of braided Gr-categories. Section 3 is devoted to a brief summary of finite quasi-quantum groups of diagonal type. In Section 4, we present a first attempt to the classification problem of finite quasi-quantum groups of nondiagonal type. Finally, in Section 5 we provide some further research problems. Throughout the paper, $\Bbbk$ is an algebraically closed field with characteristic zero and all linear spaces are over $\Bbbk$. In accordance with our previous works [**25**, **26**, **29**, **31**], we only work on pointed coquasi-Hopf algebras. By taking linear dual, one has the version for elementary quasi-Hopf algebras. All the results mentioned in this paper can be found in [**27**, **28**, **32**, **33**]. In most cases, technical computations and long proofs are omitted. The interested reader is referred to *ibid.* for details.

2. Normalized cocycles on finite abelian groups and braided Gr-categories

In this section, we will present unified formulas for normalized cocycles of all degrees on finite abelian groups. As applications of the unified formulas of normalized 3-cocycles, we provide an explicit description of braided Gr-categories which are a simplest class of pointed finite tensor categories. For later applications, we also recall the definition of abelian 3-cocycles and their resolutions. The main results are from [**28**, **30**, **32**]. Throughout this section, we will use freely the concepts and notations about group cohomology in the book [**35**].

2.1. Normalized cocycles on finite abelian groups. Let G be a group and $(B_\bullet, \partial_\bullet)$ be its normalized bar resolution. Applying $\mathrm{Hom}_{\mathbb{Z}G}(-, \Bbbk^*)$ one gets a complex $(B_\bullet^*, \partial_\bullet^*)$. Denote the group of normalized k-cocycles by $Z^k(G, \Bbbk^*)$, which is $\mathrm{Ker}\, \partial_k^*$. In general, it is hard to determine $Z^n(G, \Bbbk^*)$ directly as the normalized bar resolution is far too large.

Our approach of formulating the normalized cocycles is straightforward and elementary. First we construct a Koszul-like resolution of a finite abelian group G by tensoring the minimal resolutions of cyclic factors of G and give a complete set of representatives of cocycles for this resolution. Then we construct a chain map from the normalized bar resolution to this Koszul-like resolution. Finally we get the desired explicit and unified formulas of normalized cocycles on G by pulling back those on the Koszul-like resolution along the chain map.

In the following we record only the formulas of normalized cocycles. The interested reader is referred to [**32**] for more details. Without loss of generality, we can assume that $G = \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_n}$ where $m_i | m_{i+1}$ for all $1 \le i \le n-1$. Let g_i be a fixed generator of $\mathbb{Z}_{m_i}$, $1 \le i \le n$. Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $\alpha_u = (\alpha_{u1}, \ldots, \alpha_{un})$ for $1 \le u \le k$, where each $\alpha_{ij} \in [0, m_j)$ and is viewed as an integer modulo m_j for all $1 \le i \le k$. For α_u, $1 \le u \le k$, we denote

$$g^{\alpha_u} = g_1^{\alpha_{u1}} \cdots g_n^{\alpha_{un}}.$$

For each positive integer m, fix a primitive m-th root of unity ζ_m. Let $1 \leq a < b \leq k, 1 \leq r \leq n$ be integers. Then for a fixed α we can define

$$\eta^{\alpha}_{r,[a,b]} := \begin{cases} [\frac{\alpha_{br}+\alpha_{b-1,r}}{m_r}] \cdots [\frac{\alpha_{a+1,r}+\alpha_{ar}}{m_r}], & b-a \text{ odd;} \\ [\frac{\alpha_{br}+\alpha_{b-1,r}}{m_r}] \cdots [\frac{\alpha_{a+2,r}+\alpha_{a+1,r}}{m_r}]\alpha_{ar}, & b-a \text{ even.} \end{cases}$$

Here by $[\frac{s}{t}]$ we denote the integer part of $\frac{s}{t}$ for any positive integers s and t. Let

$$\mathcal{R}_l := \{(r_1, r_2, \cdots, r_l) | 1 \leq r_1 < r_2 < \cdots < r_l \leq n\},$$

$$\Lambda_l := \{(\lambda_1, \lambda_2, \cdots, \lambda_l) | \lambda_1 + \lambda_2 + \cdots \lambda_l = k,\ \lambda_1 \text{ odd},\ \lambda_i \geq 1 \text{ for } 1 \leq i \leq l\}.$$

Let Γ be the set of all sequences of the form

$$(a_{r_1^{\lambda_1} \cdots r_l^{\lambda_l}})_{(r_1, \cdots, r_l) \in \mathcal{R}_l,\ (\lambda_1, \cdots, \lambda_l) \in \Lambda_l,\ \mathbf{1} \leq l \leq k}$$

such that $0 \leq a_{r_1^{\lambda_1} \cdots r_l^{\lambda_l}} \leq m_{r_1}$ for $(r_1, \cdots, r_l) \in \mathcal{R}_l, (\lambda_1, \cdots, \lambda_l) \in \Lambda_l, 1 \leq l \leq k$.

Then we have the following

PROPOSITION 2.1. [**32**, Corollary 2.5] *For each sequence* $a = (a_{r_1^{\lambda_1} \cdots r_l^{\lambda_l}})_{(r_1, \cdots, r_l) \in \mathcal{R}_l,\ (\lambda_1, \cdots, \lambda_l) \in \Lambda_l,\ \mathbf{1} \leq l \leq k}$ *in* Γ, *we can define a* k*-cochains* $\omega \in \mathrm{Hom}_{\mathbb{Z}G}(B_k, \Bbbk^*)$ *by*

$$\omega_a([g^{\alpha_1}, \ldots, g^{\alpha_k}]) \tag{2.1}$$
$$= \prod_{l=1}^{k} \prod_{\substack{(r_1, \cdots, r_l) \in \mathcal{R}_l \\ (\lambda_1, \cdots, \lambda_l) \in \Lambda_l}} \zeta_{m_{r_1}}^{(-1)^{\sum_{1 \leq i < j \leq l} \lambda_i \lambda_j} \eta^{\alpha}_{r_1,[a_1,b_1]} \cdots \eta^{\alpha}_{r_l,[a_l,b_l]} a_{r_1^{\lambda_1} \cdots r_l^{\lambda_l}}}$$

where $a_u = \sum_{i=u+1}^{l} \lambda_i + 1, \quad b_u = \sum_{i=u}^{l} \lambda_i$ *for* $(\lambda_1, \cdots, \lambda_l) \in \Lambda_l$, $1 \leq u \leq l$. *The set* $\{\omega_a | a \in \Gamma\}$ *is a complete set of representatives of* k*-cocycles of the complex* $(B^*_\bullet, \partial^*_\bullet)$.

2.2. Braided Gr-categories. The monoidal category of finite dimensional vector spaces graded by a group G, with the usual tensor product and associativity constraint given by a 3-cocycle ω is denoted by Vec^{ω}_G. Such a monoidal category is called a linear Gr-category. The terminology goes back to Hoàng Xuân Sính [**24**], a student of Grothendieck. The aim of this subsection is to give a complete description to all braided linear Gr-categories with a help of the explicit formulas of normalized 3-cocycles.

Recall that the category Vec_G of finite-dimensional G-graded vector spaces has simple objects $\{S_g | g \in G\}$ where $(S_g)_h = \delta_{g,h}\Bbbk$, $\forall h \in G$. The tensor product is given by $S_g \otimes S_h = S_{gh}$, and S_1 (1 is the identity of G) is the unit object. Without loss of generality we may assume that the left and right unit constraints are identities. If a is an associativity constraint on Vec_G, then it is given by $a_{S_f,S_g,S_h} = \omega(f,g,h)\,\mathrm{id}$, where $\omega : G \times G \times G \to \Bbbk^*$ is a function. The pentagon axiom and the triangle axiom give

$$\omega(ef,g,h)\omega(e,f,gh) = \omega(e,f,g)\omega(e,fg,h)\omega(f,g,h),$$
$$\omega(f,1,g) = 1,$$

which say exactly that ω is a normalized 3-cocycle on G. Note that cohomologous cocycles define equivalent monoidal structures, therefore the classification of monoidal structures on Vec_G is equivalent to determining a complete set of representatives of normalized 3-cocycles on G.

Keep the notations of Subsection 2.1. One can easily give the explicit formulas of normalized 3-cocycle on G by (2.1). Define $\mathcal{A}$ to be the set of all sequences like

$$(a_1, \dots, a_l, \dots, a_n, a_{12}, \dots, a_{ij}, \dots, a_{n-1,n}, a_{123}, \dots, a_{rst}, \dots, a_{n-2,n-1,n}) \tag{2.2}$$

such that $0 \le a_l < m_l,\ 0 \le a_{ij} < m_i,\ 0 \le a_{rst} < m_r$ for $1 \le l \le n,\ 1 \le i < j \le n,\ 1 \le r < s < t \le n$ where a_{ij} and a_{rst} are ordered by the lexicographic order. In the following, the sequence (2.2) is denoted by $\underline{\mathbf{a}}$ for short. In the special case $k = 3$, if we abbreviate a_{r^3}, $a_{r^1s^2}$ and $a_{r^1s^1t^1}$ in formula (2.1) by a_r, a_{rs} and a_{rst} respectively, then (2.1) becomes

$$\begin{aligned} &\Phi_{\underline{\mathbf{a}}} : B_3 \longrightarrow \Bbbk^* \\ &[g_1^{i_1} \cdots g_n^{i_n}, g_1^{j_1} \cdots g_n^{j_n}, g_1^{k_1} \cdots g_n^{k_n}] \\ &\mapsto \prod_{r=1}^{n} \zeta_{m_r}^{a_r i_r [\frac{j_r+k_r}{m_r}]} \prod_{1 \le r < s \le n} \zeta_{m_r}^{a_{rs} k_r [\frac{i_s+j_s}{m_s}]} \prod_{1 \le r < s < t \le n} \zeta_{m_r}^{-a_{rst} k_r j_s i_t} \end{aligned} \tag{2.3}$$

where $0 \le a_r, a_{rs}, a_{rst} < m_r$. This is a complete set of representatives of normalized 3-cocycles on G.

Now we consider the braided structures on a linear Gr-category Vec_G^ω. Recall that a braiding in Vec_G^ω is a commutativity constraint $c : \otimes \to \otimes^{\mathrm{op}}$ satisfying the hexagon axiom. Note that c is given by $c_{S_x,S_y} = \mathcal{R}(x,y)\,\mathrm{id}$, where $\mathcal{R} : G \times G \to \Bbbk^*$ is a function, and the hexagon axiom of c says that

$$\frac{\mathcal{R}(xy,z)}{\mathcal{R}(x,z)\mathcal{R}(y,z)} \frac{\omega(x,z,y)}{\omega(x,y,z)\omega(z,x,y)} = 1 = \frac{\mathcal{R}(x,yz)}{\mathcal{R}(x,y)\mathcal{R}(x,z)} \frac{\omega(x,y,z)\omega(y,z,x)}{\omega(y,x,z)} \tag{2.4}$$

for all $x, y, z \in G$.

In other words, $\mathcal{R}$ is a quasi-bicharacter of G with respect to ω. Therefore, the classification of braidings in Vec_G^ω is equivalent to determining all the quasi-bicharacters of G with respect to ω. Clearly, any quasi-bicharacter $\mathcal{R}$ is uniquely determined by the following values:

$$r_{ij} := \mathcal{R}(g_i, g_j), \qquad \text{for all } 1 \le i,\ j \le n.$$

Proposition 2.2. [**32**, Proposition 3.2] *Let $r_{ij} \in \Bbbk^*$ for $1 \le i,\ j \le n$. Then there is a quasi-bicharacter $\mathcal{R}$ with respect to ω satisfying $\mathcal{R}(g_i, g_j) = r_{ij}$ if and only if the following equations are satisfied:*

$$\begin{aligned} r_{ii}^{m_i} = \zeta_{m_i}^{a_i} = \zeta_{m_i}^{-a_i}, \qquad & \textit{for } 1 \le i \le n, \\ r_{ij}^{m_i} = r_{ji}^{m_i} = 1, \quad a_{ij} = 0, \qquad & \textit{for } 1 \le i < j \le n, \\ a_{rst} = 0, \qquad & \textit{for } 1 \le r < s < t \le n. \end{aligned}$$

2.3. Abelian 3-cocycles. In order to define abelian 3-cocycles, we need to recall first twisted quantum doubles [**12**]. By definition, the twisted quantum double $D^\Phi(G)$ of G with respect to the 3-cocycle Φ over G is the semisimple quasi-Hopf algebra with underlying vector space $(\Bbbk G)^* \otimes \Bbbk G$ in which multiplication, comultiplication Δ, associator ϕ, counit ε, antipode $\mathcal{S}$, α and β are given by

$$
\begin{aligned}
&(e(g)\otimes x)(e(h)\otimes y) = \theta_g(x,y)\delta_{g,h}e(g)\otimes xy,\\
&\Delta(e(g)\otimes x) = \sum_{hk=g}\gamma_x(h,k)e(h)\otimes x\otimes e(k)\otimes x,\\
&\phi = \sum_{g,h,k\in G}\Phi(g,h,k)^{-1}e(g)\otimes 1\otimes e(h)\otimes 1\otimes e(k)\otimes 1,\\
&\mathcal{S}(e(g)\otimes x) = \theta_{g^{-1}}(x,x^{-1})^{-1}\gamma_x(g,g^{-1})^{-1}e(g^{-1})\otimes x^{-1},\\
&\varepsilon(e(g)\otimes x) = \delta_{g,1},\quad \alpha = 1,\quad \beta = \sum_{g\in G}\Phi(g,g^{-1},g)e(g)\otimes 1,
\end{aligned}
$$

where $\{e(g)|g\in G\}$ is the dual basis of $\{g|g\in G\}$, $\delta_{x,y}$ is the Kronecker delta, and

$$
\begin{aligned}
\theta_g(x,y) &= \frac{\Phi(g,x,y)\Phi(x,y,g)}{\Phi(x,g,y)},\\
\gamma_g(x,y) &= \frac{\Phi(x,y,g)\Phi(g,x,y)}{\Phi(x,g,y)}
\end{aligned}
$$

for any $x,y,g\in G$.

Definition 2.3. A 3-cocycle Φ over G is called abelian if $D^{\Phi}(G)$ is a commutative algebra.

With a help of our explicit formulas of normalized 3-cocycles, one can easily single out the abelian ones. Let $\mathbb{G}$ be a finite abelian group. So $\mathbb{G}\cong\mathbb{Z}_{\mathbb{m}_1}\times\cdots\times\mathbb{Z}_{\mathbb{m}_n}$ with $\mathbb{m}_j\in\mathbb{N}$ for $1\le j\le n$ and $\mathbb{m}_i|\mathbb{m}_{i+1}$ for all $1\le i\le n-1$. Let $\mathbb{g}_i$ be a generator of $\mathbb{Z}_{\mathbb{m}_i}$. Suppose Φ is a normalized 3-cocycle on $\mathbb{G}$. Thanks to (2.3), we may assume that $\Phi=\Phi_{\underline{\mathbf{a}}}$ for some $\mathbf{a}\in A$. Then we have the following numerical description of abelian 3-cocycles.

Proposition 2.4. *The 3-cocycle $\Phi_{\underline{\mathbf{a}}}$ is abelian if and only if $a_{rst}=0$ for all $1\le r<s<t\le n$.*

Proof. "$\Leftarrow$:" If all $a_{rst}=0$, then by (2.3) it is not hard to find that

$$\Phi_{\underline{\mathbf{a}}}(x,y,z)=\Phi_{\underline{\mathbf{a}}}(x,z,y)$$

for $x,y,z\in\mathbb{G}$. From this, we can find that

$$\theta_g(x,y)=\theta_g(y,x)$$

for $g,x,y\in\mathbb{G}$, which implies that $D^{\Phi_{\underline{\mathbf{a}}}}(\mathbb{G})$ is commutative.

"$\Rightarrow$:" If $a_{rst}\neq 0$ for some $r<s<t$. Through direct computations, we have

$$\theta_{\mathbb{g}_r}(\mathbb{g}_s,\mathbb{g}_t)=1,\qquad \theta_{\mathbb{g}_r}(\mathbb{g}_t,\mathbb{g}_s)=\zeta_{\mathbb{m}_r}^{-a_{rst}}.$$

This implies that

$$(e(\mathbb{g}_r)\otimes\mathbb{g}_s)(e(\mathbb{g}_r)\otimes\mathbb{g}_t)\neq(e(\mathbb{g}_r)\otimes\mathbb{g}_t)(e(\mathbb{g}_r)\otimes\mathbb{g}_s).$$

□

2.4. Resolution. Let $\mathbb{G} = \mathbb{Z}_{\mathbbm{m}_1} \times \cdots \times \mathbb{Z}_{\mathbbm{m}_n}$ as before and $\Phi_{\underline{\mathbf{a}}}$ be an abelian 3-cocycle of $\mathbb{G}$. Then $\Phi_{\underline{\mathbf{a}}}$ can be "resolved" in a slightly bigger abelian group G. More precisely, take $G = \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_n}$ for $m_i = \mathbbm{m}_i^2$ $(1 \leq i \leq n)$. As before, let $\mathbbm{g}_i$ (resp. g_i) be a generator of $\mathbb{Z}_{\mathbbm{m}_i}$ (resp. $\mathbb{Z}_{m_i}$) for $1 \leq i \leq n$. Using such notations, we have a canonical group epimorphism:

$$\pi : G \to \mathbb{G}, \quad g_i \mapsto \mathbbm{g}_i \ (1 \leq i \leq n).$$

By pulling back the 3-cocycles on $\mathbb{G}$ along π, we get 3-cocycles

$$\pi^*(\Phi_{\underline{\mathbf{a}}}) : G \times G \times G \to k^*, \quad (g, h, z) \mapsto \Phi_{\underline{\mathbf{a}}}(\pi(g), \pi(h), \pi(z)), \quad g, h, z \in G$$

on G. Then $\pi^*(\Phi_{\underline{\mathbf{a}}})$ is in fact a coboundary. To see this clearly, consider the following map

(2.5)

$$J_{\underline{\mathbf{a}}} : G \times G \to k^*; \quad (g_1^{x_1} \cdots g_n^{x_n}, g_1^{y_1} \cdots g_n^{y_n}) \mapsto \prod_{l=1}^{n} \zeta_{m_l}^{a_l x_l (y_l - y_l')} \prod_{1 \leq s < t \leq n} \zeta_{\mathbbm{m}_s \mathbbm{m}_t}^{a_{st} x_t (y_s - y_s')}$$

where y_i' is the remainder of y_i divided by $\mathbbm{m}_i$ for $1 \leq i \leq n$. Here for simple, we just take $\zeta_t = e^{\frac{2\pi i}{t}}$ for $t \in \mathbb{N}$. Thus, we have

Proposition 2.5. *The differential of $J_{\underline{\mathbf{a}}}$ equals to $\pi^*(\Phi_{\underline{\mathbf{a}}})$, that is*

$$\partial(J_{\underline{\mathbf{a}}}) = \pi^*(\Phi_{\underline{\mathbf{a}}}).$$

Proof.

$$\begin{aligned}
&\partial(J_{\underline{\mathbf{a}}})(g_1^{i_1} \cdots g_n^{i_n}, g_1^{j_1} \cdots g_n^{j_n}, g_1^{k_1} \cdots g_n^{k_n}) \\
&= \frac{J_{\underline{\mathbf{a}}}(g_1^{j_1} \cdots g_n^{j_n}, g_1^{k_1} \cdots g_n^{k_n}) J_{\underline{\mathbf{a}}}(g_1^{i_1} \cdots g_n^{i_n}, g_1^{j_1+k_1} \cdots g_n^{j_n+k_n})}{J_{\underline{\mathbf{a}}}(g_1^{i_1+j_1} \cdots g_n^{i_n+j_n}, g_1^{k_1} \cdots g_n^{k_n}) J_{\underline{\mathbf{a}}}(g_1^{i_1} \cdots g_n^{i_n}, g_1^{j_1} \cdots g_n^{j_n})} \\
&= \frac{\prod_{l=1}^{n} \zeta_{m_l}^{a_l j_l (k_l - k_l')} \prod_{1 \leq s < t \leq n} \zeta_{\mathbbm{m}_s \mathbbm{m}_t}^{a_{st} j_t (k_s - k_s')} \prod_{l=1}^{n} \zeta_{m_l}^{a_l i_l (j_l + k_l - (j_l + k_l)')} \prod_{1 \leq s < t \leq n} \zeta_{\mathbbm{m}_s \mathbbm{m}_t}^{a_{st} i_t (j_s + k_s - (j_s + k_s)')}}{\prod_{l=1}^{n} \zeta_{m_l}^{a_l (i_l + j_l)(k_l - k_l')} \prod_{1 \leq s < t \leq n} \zeta_{\mathbbm{m}_s \mathbbm{m}_t}^{a_{st} (i_t + j_t)(k_s - k_s')} \prod_{l=1}^{n} \zeta_{m_l}^{a_l i_l (j_l - j_l')} \prod_{1 \leq s < t \leq n} \zeta_{\mathbbm{m}_s \mathbbm{m}_t}^{a_{st} i_t (j_s - j_s')}} \\
&= \prod_{l=1}^{n} \zeta_{m_l}^{a_l i_l (j_l' + k_l' - (j_l + k_l)')} \prod_{1 \leq s < t \leq n} \zeta_{\mathbbm{m}_s \mathbbm{m}_t}^{a_{st} i_t (j_s' + k_s' - (j_s + k_s)')} \\
&= \prod_{l=1}^{n} \zeta_{\mathbbm{m}_l}^{a_l i_l' [\frac{j_l' + k_l'}{\mathbbm{m}_l}]} \prod_{1 \leq s < t \leq n} \zeta_{\mathbbm{m}_t}^{a_{st} i_t' [\frac{j_s' + k_s'}{\mathbbm{m}_s}]} \\
&= \pi^*(\Phi_{\underline{\mathbf{a}}})(g_1^{i_1} \cdots g_n^{i_n}, g_1^{j_1} \cdots g_n^{j_n}, g_1^{k_1} \cdots g_n^{k_n}).
\end{aligned}$$

□

However, there is no chance to resolve non-abelian 3-cocycles on bigger abelian groups.

Proposition 2.6. [**27**, Proposition 3.17] *Let $\Phi_{\underline{\mathbf{a}}}$ be a non-abelian 3-cocycle on $\mathbb{G}$ and G be an arbitrary finite abelian group with a group epimorphism $\pi \colon G \twoheadrightarrow \mathbb{G}$. Then $\pi^*(\Phi_{\underline{\mathbf{a}}})$ is not a coboundary on G.*

In other words, for a 3-cocycle on a finite abelian group, it is abelian if and only if it is resolvable. We remark that abelian 3-cocycles are called trivializable in [**7**, **11**] and some similar equivalent conditions are provided therein.

3. Finite quasi-quantum groups of diagonal type: A brief summary

In this section, we give a brief summary on a classification of finite-dimensional coradically graded pointed coquasi-Hopf algebras of diagonal type. The results presented here are included in [**27**, **28**].

Let $(\mathbb{M}, \Phi)$ be a finite-dimensional coradically graded pointed coquasi-Hopf algebra. Recall from Introduction that $\mathbb{M}$ can be factorized as $R\#\Bbbk\mathbb{G}$ with $\mathbb{G}$ a finite group and R a Hopf algebra in the twisted Yetter-Drinfeld category ${}_{\mathbb{G}}^{\mathbb{G}}\mathcal{YD}^{\Phi}$. If $\mathbb{G}$ is abelian, then $(\Bbbk\mathbb{G}, \Phi)$ is completely clear by Section 2. So the study of $\mathbb{M}$ is reduced to that of R. In accordance with the theory of finite-dimensional pointed Hopf algebras, the key here is the Nichols algebras in twisted Yetter-Drinfeld categories of finite abelian groups. We call the Nichols algebra $\mathcal{B}(V)$ of an object $V \in {}_{\mathbb{G}}^{\mathbb{G}}\mathcal{YD}^{\Phi}$ is of diagonal type, if V itself is of diagonal type (that is, a direct sum of 1-dimensional simple objects in ${}_{\mathbb{G}}^{\mathbb{G}}\mathcal{YD}^{\Phi}$.) Similarly, we call $\mathbb{M}$ diagonal if R is diagonal in ${}_{\mathbb{G}}^{\mathbb{G}}\mathcal{YD}^{\Phi}$. Thanks to Subsection 2.3, all simple objects of ${}_{\mathbb{G}}^{\mathbb{G}}\mathcal{YD}^{\Phi}$ are 1-dimensional if and only if both $\mathbb{G}$ and Φ are abelian. Moreover, in this case the Nichols algebra $\mathcal{B}(V)$ of $V \in {}_{\mathbb{G}}^{\mathbb{G}}\mathcal{YD}^{\Phi}$ can be transformed to a usual Nichols algebra as the abelian 3-cocycle Φ is resolvable. With this definite connection, the theory of finite-dimensional pointed Hopf algebras over abelian groups can be applied to our quasi situation.

Thus, in order to classify finite-dimensional coradically graded pointed coquasi-Hopf algebras of diagonal type, the main task is to give a classification of the Nichols algebras of diagonal type with arithmetic root system in ${}_{\mathbb{G}}^{\mathbb{G}}\mathcal{YD}^{\Phi}$. The idea to realize our purpose consists of five steps. Firstly, we can assume that the support group of $\mathcal{B}(V)$ is $\mathbb{G}$, and from this assumption we can prove that Φ must be an abelian 3-cocycle over $\mathbb{G}$. Secondly, we will develop a technique to change the base group from $\mathbb{G}$ to a bigger one G together with a group epimorphism $\pi : G \to \mathbb{G}$. Thirdly, we will show that any Nichols algebra $\mathcal{B}(V)$ in ${}_{\mathbb{G}}^{\mathbb{G}}\mathcal{YD}^{\Phi}$ is isomorphic to a Nichols algebra in ${}_{G}^{G}\mathcal{YD}^{\pi^*(\Phi)}$, which is thus twist equivalent to a usual Nichols algebra by Proposition 2.5. Fourthly, we want to get a return ticket, that is, we will give a sufficient and necessary condition to determine when a Nichols algebra in ${}_{\mathbb{G}}^{\mathbb{G}}\mathcal{YD}^{\pi^*(\Phi)}$ is isomorphic to one in ${}_{\mathbb{G}}^{\mathbb{G}}\mathcal{YD}^{\Phi}$. Finally, combining these results and Heckenberger's classification of arithmetic root systems, we obtain the classification of Nichols algebras of diagonal type with arithmetic root system in ${}_{\mathbb{G}}^{\mathbb{G}}\mathcal{YD}^{\Phi}$.

3.1. Yetter-Drinfeld modules over $(\Bbbk\mathbb{G}, \Phi)$ and Nichols algebras.

There are general definitions of Yetter-Drinfeld modules over coquasi-Hopf algebras and Nichols algebras of braided tensor categories. For our purpose, it is enough to recall only the definitions of Yetter-Drinfeld modules over coquasi-Hopf algebras of form $(\Bbbk\mathbb{G}, \Phi)$ with $\mathbb{G}$ an abelian group and Nichols algebra in ${}_{\mathbb{G}}^{\mathbb{G}}\mathcal{YD}^{\Phi}$.

Assume that V is a left $\Bbbk\mathbb{G}$-comudule with comodule structure map $\delta_L : V \to \Bbbk\mathbb{G} \otimes V$. Define ${}^{g}V := \{v \in V | \delta_L(v) = g \otimes v\}$ and thus $V = \bigoplus_{g \in G} {}^{g}V$. Here we call g the degree of the elements in ${}^{g}V$ and denote by $\deg v = g$ for $v \in {}^{g}V$. For the 3-cocycle Φ on $\mathbb{G}$ and any $g \in \mathbb{G}$, define

$$\widetilde{\Phi}_g : \mathbb{G} \times \mathbb{G} \to \Bbbk^*, \quad (e, f) \mapsto \frac{\Phi(g, e, f)\Phi(e, f, g)}{\Phi(e, g, f)}. \tag{3.1}$$

Direct computation shows that

$$\widetilde{\Phi}_g \in \mathbb{Z}^2(\mathbb{G}, \Bbbk^*).$$

Definition 3.1. A left $\Bbbk\mathbb{G}$-comudule V is a left-left Yetter-Drinfeld module over $(\Bbbk\mathbb{G}, \Phi)$ if each ${}^gV$ is a projective $\mathbb{G}$-representation with respect to the 2-cocycle $\widetilde{\Phi}_g$, namely the $\mathbb{G}$-action $\triangleright$ on ${}^gV$ satisfies

$$e \triangleright (f \triangleright v) = \widetilde{\Phi}_g(e, f)(ef) \triangleright v, \quad \forall e, f \in G,\ v \in {}^gV. \tag{3.2}$$

The category of left-left Yetter-Drinfeld modules is denoted by ${}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi}$. Similarly, one can define left-right, right-left and right-right Yetter-Drinfeld modules over $(\Bbbk\mathbb{G}, \Phi)$. As the familiar Hopf case, ${}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi}$ is a braided tensor category. More precisely, for any $M, N \in {}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi}$, the structure maps of $M \otimes N$ as a left-left Yetter-Drinfeld module are given by

$$\delta_L(m_g \otimes n_h) := gh \otimes m_g \otimes n_h, \quad x \triangleright (m_g \otimes n_h) := \widetilde{\Phi}_x(g, h) x \triangleright m_g \otimes x \triangleright n_h \tag{3.3}$$

for all $x, g, h \in \mathbb{G}$ and $m_g \in {}^gM,\ n_h \in {}^hN$. The associativity constraint a and the braiding c of ${}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi}$ are given respectively by

$$a((u_e \otimes v_f) \otimes w_g) = \Phi(e, f, g)^{-1} u_e \otimes (v_f \otimes w_g) \tag{3.4}$$

$$c(u_e \otimes v_f) = e \triangleright v_f \otimes u_e \tag{3.5}$$

for all $e, f, g \in \mathbb{G},\ u_e \in {}^eU,\ v_f \in {}^fV,\ w_g \in {}^gW$ and $U, V, W \in {}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi}$.

Nichols algebras can be defined by various equivalent ways, see for example [**2**]. Here we adopt the defining method in terms of the universal property. Roughly, Nichols algebras are the analogue of the usual symmetric algebras in more general braided tensor categories.

Let V be a nonzero object in ${}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi}$. By $T_\Phi(V)$ we denote the tensor algebra in ${}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi}$ generated freely by V. It is clear that $T_\Phi(V)$ is isomorphic to $\bigoplus_{n\geq 0} V^{\otimes \overrightarrow{n}}$ as a linear space, where $V^{\otimes \overrightarrow{n}}$ means

$$\underbrace{(\cdots((}_{n-1} V \otimes V) \otimes V) \cdots \otimes V).$$

This induces a natural $\mathbb{N}$-graded structure on $T_\Phi(V)$. Define a comultiplication on $T_\Phi(V)$ by $\Delta(X) = X \otimes 1 + 1 \otimes X,\ \forall X \in V$, a counit by $\varepsilon(X) = 0$, and an antipode by $S(X) = -X$. These provide a graded Hopf algebra structure on $T_\Phi(V)$ in the braided tensor category ${}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi}$.

Definition 3.2. The Nichols algebra $\mathcal{B}(V)$ of V is defined to be the quotient Hopf algebra $T_\Phi(V)/I$ in ${}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi}$, where I is the unique maximal graded Hopf ideal generated by homogeneous elements of degree greater than or equal to 2.

To stress that our Nichols algebras may be nonassociative in some occasions, we will call an associative Nichols algebra, e.g. $\mathcal{B}(V) \in {}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}$, a usual Nichols algebra. The following definition is used widely in the rest of this paper.

Definition 3.3. Let $V \in {}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi}$ be a Yetter-Drinfeld module of diagonal type and $\{X_i | 1 \leq i \leq n\}$ be a standard basis of V, and $\{g_i | 1 \leq i \leq n\}$ be the corresponding degrees, that is $\delta_L(X_i) = g_i \otimes X_i$ for $1 \leq i \leq n$. Then we call the subgroup $\mathbb{G}' = \langle g_1, \cdots, g_n\rangle$ generated by $g_1, \ldots, g_n$ the support group of V, which is denoted by $\mathbb{G}_V$.

It is obvious that the definition does not depend on the choices of the standard bases and

$$\mathbb{G}_V = \mathbb{G}_{\mathcal{B}(V)} = \mathbb{G}_{T_\Phi(V)}.$$

The twisting process for coquasi-Hopf algebras can be transferred to Nichols algebras directly. In fact, let $(V, \triangleright, \delta_L) \in {}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi}$, and J a 2-cochain of $\mathbb{G}$. Then we can define a new action $\triangleright_J$ of $\mathbb{G}$ over V by

$$g \triangleright_J X = \frac{J(g,x)}{J(x,g)} g \triangleright X \tag{3.6}$$

for $X \in V$ and $g \in \mathbb{G}$. We denote $(V, \triangleright_J, \delta_L)$ by V^J and by definition we have $V^J \in {}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi * \partial(J)}$. Moreover there is a tensor equivalence $(F_J, \varphi_0, \varphi_2) : {}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi} \to {}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi * \partial(J)}$ which takes V to V^J and

$$\varphi_2(U, V) : (U \otimes V)^J \to U^J \otimes V^J, \quad Y \otimes Z \mapsto J(y,z)^{-1} Y \otimes Z$$

for $Y \in U,\ Z \in V$.

Let $\mathcal{B}(V)$ be a usual Nichols algebra in ${}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}$, then it is clear that $\mathcal{B}(V)^J$ is a Hopf algebra in ${}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\partial J}$ with multiplication $\circ$ determined by

$$X \circ Y = J(x,y) XY \tag{3.7}$$

for all homogenous elements $X, Y \in \mathcal{B}(V)$, here $x = \deg X, y = \deg Y$. Using the same terminology as (co)quasi-Hopf algebras, we call $\mathcal{B}(V)$ and $\mathcal{B}(V)^J$ are twist equivalent. The following fact is obvious, but important for our exposition.

LEMMA 3.4. *The twisting $\mathcal{B}(V)^J$ of $\mathcal{B}(V)$ is a Nichols algebra in ${}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\partial J}$ and $\mathcal{B}(V)^J \cong \mathcal{B}(V^J)$.*

3.2. Pointed coquasi-Hopf algebras and related notions. Recall that a coquasi-Hopf algebra $\mathbb{M}$ is said to be pointed if the underlying coalgebra is so. Given a pointed coquasi-Hopf algebra $(\mathbb{M}, \Delta, \varepsilon, \mathrm{M}, \mu, \Phi, \mathcal{S}, \alpha, \beta)$, let $\{\mathbb{M}_n\}_{n \geq 0}$ be its coradical filtration, and

$$\operatorname{gr} \mathbb{M} = \mathbb{M}_0 \oplus \mathbb{M}_1/\mathbb{M}_0 \oplus \mathbb{M}_2/\mathbb{M}_1 \oplus \cdots$$

the corresponding coradically graded coalgebra. Then $\operatorname{gr} \mathbb{M}$ is a coradically graded coquasi-Hopf algebra. The corresponding graded associator $\operatorname{gr} \Phi$ satisfies $\operatorname{gr} \Phi(\bar{a}, \bar{b}, \bar{c}) = 0$ for all homogeneous $\bar{a}, \bar{b}, \bar{c} \in \operatorname{gr} \mathbb{M}$ unless they all lie in $\mathbb{M}_0$. Similar condition holds for $\operatorname{gr} \alpha$ and $\operatorname{gr} \beta$. In particular, $\mathbb{M}_0$ is a coquasi-Hopf subalgebra and it turns out to be the coquasi-Hopf algebra $(\Bbbk\mathbb{G}, \operatorname{gr} \Phi)$ for $\mathbb{G} = \mathbb{G}(\mathbb{M})$, the set of group-like elements of $\mathbb{M}$. We call a pointed Majid algebra $\mathbb{M}$ *graded* if $\mathbb{M} \cong \operatorname{gr} \mathbb{M}$ as Majid algebras.

DEFINITION 3.5. Let $(\mathbb{M}, \Delta, \varepsilon, \mathrm{M}, \mu, \Phi, \mathcal{S}, \alpha, \beta)$ be a coquasi-Hopf algebra. A convolution-invertible linear map

$$J : \ \mathbb{M} \otimes \mathbb{M} \to \Bbbk$$

is called a twisting (or gauge transformation) on $\mathbb{M}$ if

$$J(h, 1) = \varepsilon(h) = J(1, h)$$

for all $h \in \mathbb{M}$.

Given a coquasi-Hopf algebra $\mathbb{M}$ and a twisting J, then one can construct a new coquasi-Hopf algebra $\mathbb{M}^J$ as follows: $\mathbb{M}^J = \mathbb{M}$ as a coalgebra and the multiplication "$\circ$" on $\mathbb{M}^J$ is given by

$$a \circ b := J(a_1, b_1) a_2 b_2 J^{-1}(a_3, b_3) \tag{3.8}$$

for all $a, b \in \mathbb{M}$. The associator Φ^J and the quasi-antipode $(\mathcal{S}^J, \alpha^J, \beta^J)$ are given as:

$$\Phi^J(a,b,c) = J(b_1,c_1)J(a_1,b_2c_2)\Phi(a_2,b_3,c_3)J^{-1}(a_3b_4,c_4)J^{-1}(a_4,b_5),$$

$$\mathcal{S}^J = \mathcal{S}, \quad \alpha^J(a) = J^{-1}(\mathcal{S}(a_1),a_3)\alpha(a_2), \quad \beta^J(a) = J(a_1,\mathcal{S}(a_3))\beta(a_2)$$

for all $a, b, c \in \mathbb{M}$.

Definition 3.6. Two coquasi-Hopf algebras $\mathbb{M}_1$ and $\mathbb{M}_2$ are called twist equivalent if there is a twisting J on $\mathbb{M}_1$ such that $\mathbb{M}_1^J \cong \mathbb{M}_2$ as coquasi-Hopf algebras. Denote $\mathbb{M}_1 \sim \mathbb{M}_2$ if $\mathbb{M}_1$ is twist equivalent to $\mathbb{M}_2$. We call a coquasi-Hopf algebra $\mathbb{M}$ genuine if it is not twist equivalent to a Hopf algebra.

Now suppose $\mathbb{M} = \sum_{i \geq 0} \mathbb{M}_i$ is a coradically graded pointed coquasi-Hopf algebra. Let $\pi: \mathbb{M} \to \mathbb{M}_0$ be the canonical projection. Then $\mathbb{M}$ is a $k\mathbb{G}$-bicomodule naturally via

$$\delta_L := (\pi \otimes \mathrm{id})\Delta, \quad \delta_R := (\mathrm{id} \otimes \pi)\Delta.$$

Thus there is a $\mathbb{G}$-bigrading on $\mathbb{M}$, that is,

$$\mathbb{M} = \bigoplus_{g,h \in \mathbb{G}} {}^g\mathbb{M}^h$$

where ${}^g\mathbb{M}^h = \{m \in \mathbb{M} | \delta_L(m) = g \otimes m, \ \delta_R(m) = m \otimes h\}$. As a convention, we only deal with homogeneous elements with respect to this $\mathbb{G}$-bigrading in this subsection. For example, whenever we write $\Delta(X) = X_1 \otimes X_2$, all X, X_1, X_2 are assumed homogeneous, and for any capital $X \in {}^g\mathbb{M}^h$, we use its lowercase x to denote gh^{-1}.

Define the coinvariant subalgebra of $\mathbb{M}$ by

$$\mathcal{R} := \{m \in \mathbb{M} | (\mathrm{id} \otimes \pi)\Delta(m) = m \otimes 1\}.$$

Clearly $1 \in \mathcal{R}$. There is a $(\Bbbk\mathbb{G}, \Phi)$-action on $\mathcal{R}$ via

$$f \triangleright X := \frac{\Phi(fg, f^{-1}, f)}{\Phi(f, f^{-1}, f)}(f \cdot X) \cdot f^{-1} \tag{3.9}$$

for all $f, g \in \mathbb{G}$ and $X \in {}^g\mathcal{R}$. Here $\cdot$ is the multiplication in $\mathbb{M}$. Then $(\mathcal{R}, \delta_L, \triangleright)$ is a left-left Yetter-Drinfeld module over $(\Bbbk\mathbb{G}, \Phi)$.

$$\mathrm{M}: \ \mathcal{R} \otimes \mathcal{R} \to \mathcal{R}, \quad (X, Y) \mapsto XY := X \cdot Y;$$
$$u\colon \Bbbk \to \mathcal{R}, \quad \lambda \mapsto \lambda 1;$$
$$\Delta_{\mathcal{R}}: \ \mathcal{R} \to \mathcal{R} \otimes \mathcal{R}, \quad X \mapsto \Phi(x_1, x_2, x_2^{-1}) X_1 \cdot x_2^{-1} \otimes X_2;$$
$$\varepsilon_{\mathcal{R}}: \ \mathcal{R} \to \Bbbk, \quad \varepsilon_{\mathcal{R}} := \varepsilon|_{\mathcal{R}};$$
$$\mathcal{S}_{\mathcal{R}}: \ \mathcal{R} \to \mathcal{R}, \quad X \mapsto \tfrac{1}{\Phi(x, x^{-1}, x)} x \cdot \mathcal{S}(X).$$

Then it is routine to verify that $(\mathcal{R}, \mathrm{M}, u, \Delta_{\mathcal{R}}, \varepsilon_{\mathcal{R}}, \mathcal{S}_{\mathcal{R}})$ is a Hopf algebra in ${}_{\mathbb{G}}^{\mathbb{G}}\mathcal{YD}^{\Phi}$.

Conversely, let H be a Hopf algebra in ${}_{\mathbb{G}}^{\mathbb{G}}\mathcal{YD}^{\Phi}$. Since H is a left $\mathbb{G}$-comodule, there is a $\mathbb{G}$-grading on H:

$$H = \bigoplus_{x \in G} {}^xH$$

where ${}^xH = \{X \in H | \delta_L(X) = x \otimes X\}$. As before, we only need to deal with $\mathbb{G}$-homogeneous elements. As a convention, homogeneous elements in H are denoted by capital letters, say $X, Y, Z, \ldots$, and the associated degrees are denoted by their lower cases, say $x, y, z, \ldots$.

For our purpose, we also assume that H is $\mathbb{N}$-graded with $H_0 = \mathbb{k}$. If $X \in H_n$, then we say that X has length n. Moreover, we assume that both gradings are compatible in the sense that

$$H = \bigoplus_{g \in \mathbb{G}} {}^g H = \bigoplus_{g \in \mathbb{G}} \bigoplus_{n \in \mathbb{N}} {}^g H_n.$$

For example, the Hopf algebra $\mathcal{R}$ in ${}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi}$ considered above satisfies these assumptions as $\mathcal{R} = \bigoplus_{i \in \mathbb{N}} \mathcal{R}_i$ is coradically graded. In this case, we call $\dim \mathcal{R}_1$ the rank of $\mathcal{R}$ and $\mathbb{M}$. For any $X \in H$, we write its comultiplication as

$$\Delta_H(X) = X_{(1)} \otimes X_{(2)}.$$

The following bosonization formulas come from [**34**], a more general version for coquasi-bialgebras can be found in [**10**].

Lemma 3.7. [**34**, Proposition 3.3] *Keep the assumptions on H as above. Define on $H \otimes \mathbb{k}\mathbb{G}$ a product by*

$$(X \otimes g)(Y \otimes h) = \frac{\Phi(xg, y, h)\Phi(x, y, g)}{\Phi(x, g, y)\Phi(xy, g, h)} X(g \triangleright Y) \otimes gh, \tag{3.10}$$

and a coproduct by

$$\Delta(X \otimes g) = \Phi(x_{(1)}, x_{(2)}, g)^{-1}(X_{(1)} \otimes x_{(2)}g) \otimes (X_{(2)} \otimes g). \tag{3.11}$$

Then $H \otimes \mathbb{k}\mathbb{G}$ becomes a graded coquasi-Hopf algebra with a quasi-antipode $(\mathcal{S}, \alpha, \beta)$ given by

$$\mathcal{S}(X \otimes g) = \frac{\Phi(g^{-1}, g, g^{-1})}{\Phi(x^{-1}g^{-1}, xg, g^{-1})\Phi(x, g, g^{-1})}(1 \otimes x^{-1}g^{-1})(\mathcal{S}_H(X) \otimes 1), \tag{3.12}$$

$$\alpha(1 \otimes g) = 1, \quad \alpha(X \otimes g) = 0, \tag{3.13}$$

$$\beta(1 \otimes g) = \Phi(g, g^{-1}, g)^{-1}, \quad \beta(X \otimes g) = 0, \tag{3.14}$$

here $g, h \in \mathbb{G}$ and X, Y are homogeneous elements of length ≥ 1.

Finally, we give the definition of connected pointed coquasi-Hopf algebras.

Definition 3.8. Suppose $\mathbb{M}$ is a pointed coquasi-Hopf algebra and R is the coinvariant subalgebra of $\mathrm{gr}(\mathbb{M})$, then we say that $\mathbb{M}$ is connected if $\mathbb{G}(\mathbb{M}) = \mathbb{G}_R$, where $\mathbb{G}_R$ is the support group of R.

3.3. Arithmetic root systems and generalized Dynkin diagrams.

Arithmetic root systems are invariants of Nichols algebras of diagonal type with certain finiteness property. A complete classification of arithmetic root systems was given by Heckenberger [**22**]. This is a crucial ingredient for the classification program of finite-dimensional pointed Hopf algebras, and turns out to be equally important in the broader situation of pointed coquasi-Hopf algebras.

Suppose $\mathcal{B}(V)$ is a usual Nichols algebra of diagonal type in ${}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}$. Let $\{X_i | 1 \leq i \leq n\}$ be a standard basis of V with $\delta_l(X_i) = h_i \otimes X_i$. The structure constants of $\mathcal{B}(V)$ are $\{q_{ij} | 1 \leq i, j \leq n\}$ such that $h_i \triangleright X_j = q_{ij}X_j$. Let $E = \{e_i | 1 \leq i \leq d\}$ be a canonical basis of $\mathbb{Z}^n$, and χ be a bicharacter of $\mathbb{Z}^n$ determined by $\chi(e_i, e_j) = q_{ij}$. As defined in [**19**, Sec.3], $\triangle^+(\mathcal{B}(V))$ is the set of degrees of the (restricted) Poincare-Birkhoff-Witt generators counted with multiplicities and $\triangle(\mathcal{B}(V)) := \triangle^+(\mathcal{B}(V)) \bigcup - \triangle^+(\mathcal{B}(V))$, which is called the root system of $\mathcal{B}(V)$.

Moreover, the triple $(\triangle(\mathcal{B}(V)), \chi, E))$ is called an arithmetic root system of $\mathcal{B}(V)$ if the corresponding Weyl groupoid $W_{\chi,E}$ is full and finite. In this case, we denote this arithmetic root system by $\triangle(\mathcal{B}(V))_{\chi,E}$ for brevity. If there is another arithmetic root system $\triangle_{\chi',E'}$, and an isomorphism $\tau : \mathbb{Z}^n \to \mathbb{Z}^n$ such that

$$\tau(E) = E', \quad \chi'(\tau(e), \tau(e)) = \chi(e, e),$$

$$\chi'(\tau(e_1), \tau(e_2))\chi'(\tau(e_2), \tau(e_1)) = \chi(e_1, e_2)\chi'(e_2, e_1),$$

then we say that $\triangle_{\chi,E}$ and $\triangle_{\chi',E'}$ are twist equivalent.

Generalized Dynkin diagrams are invariants of arithmetic root systems, and they can determine arithmetic root systems up to twist equivalence.

Definition 3.9. The generalized Dynkin diagram of an arithmetic root system $\triangle_{\chi,E}$ is a nondirected graph $\mathcal{D}_{\chi,E}$ with the following properties:

1) There is a bijective map ϕ from $I = \{1, 2, \ldots, d\}$ to the set of vertices of $\mathcal{D}_{\chi,E}$.
2) For all $1 \le i \le d$, the vertex $\phi(i)$ is labelled by q_{ii}.
3) For all $1 \le i, j \le d$, the number n_{ij} of edges between $\phi(i)$ and $\phi(j)$ is either 0 or 1. If $i = j$ or $q_{ij}q_{ji} = 1$ then $n_{ij} = 0$, otherwise $n_{ij} = 1$ and the edge is labelled by $\widetilde{q_{ij}} = q_{ij}q_{ji}$ for $1 \le i < j \le n$.

An arithmetic root system is called connected provided the corresponding generalized Dynkin diagram $\mathcal{D}_{\chi,E}$ is connected. All the connected arithmetic root systems are classified and the corresponding generalized Dynkin diagrams are listed in [**20**–**22**].

3.4. Classification results. In this subsection, we provide a classification of finite-dimensional Nichols algebras of diagonal type in ${}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi}$. This also leads to a complete classification of connected finite-dimensional pointed coquasi-Hopf algebras of diagonal type.

Since Nichols algebras in twisted Yetter-Drinfeld categories ${}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi}$ are nonassociative algebras, the structures of these algebras depend on $\mathbb{G}$ and the 3-cocycle Φ on $\mathbb{G}$. We will call $\mathbb{G}$ the base group of $\mathcal{B}(V)$. An important method is to change the base groups of Nichols algebras.

Definition 3.10. Let $\mathcal{B}(V)$ and $\mathcal{B}(U)$ be Nichols algebras in ${}^{G}_{G}\mathcal{YD}^{\Phi}$ and ${}^{H}_{H}\mathcal{YD}^{\Psi}$ respectively with $\dim V = \dim U = l$. We say $\mathcal{B}(V)$ is isomorphic to $\mathcal{B}(U)$ if there is a $\mathbb{Z}^l$-graded linear isomorphism $\mathcal{F} : \mathcal{B}(V) \to \mathcal{B}(U)$ which preserves the multiplication and comultiplication.

Lemma 3.11. [**28**, Lemma 4.4] *Suppose $V \in {}^{G}_{G}\mathcal{YD}^{\Phi}$ and $U \in {}^{H}_{H}\mathcal{YD}^{\Psi}$. Let G' and H' be support groups of V and U respectively. If there are a linear isomorphism $F : V \to U$ and a group epimorphism $f : G' \to H'$ such that:*

$$\delta \circ F = (f \otimes F) \circ \delta, \tag{3.15}$$

$$F(g \triangleright v) = f(g) \triangleright F(v), \tag{3.16}$$

$$\Phi|_{G'} = f^*\Psi|_{H'} \tag{3.17}$$

for any $g \in G'$, $v \in V$. Then $\mathcal{B}(V)$ is isomorphic to $\mathcal{B}(U)$.

If (F, f) is an isomorphism from $\mathcal{B}(V)$ to $\mathcal{B}(U)$ as in Lemma 3.11, then we say $\mathcal{B}(V)$ is isomorphic to $\mathcal{B}(U)$ through the group morphism f.

In the following of this subsection, suppose $\mathbb{G} = \mathbb{Z}_{\mathbb{m}_1} \times \cdots \times \mathbb{Z}_{\mathbb{m}_n} = \langle \mathbb{g}_1 \rangle \times \cdots \langle \mathbb{g}_n \rangle$ and $G = \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_n} = \langle g_1 \rangle \times \cdots \langle g_n \rangle$ where $m_i = \mathbb{m}_i^2$ for $1 \leq i \leq n$. Let

$$\pi : \ \Bbbk G \to \Bbbk \mathbb{G}, \quad g_i \mapsto \mathbb{g}_i, \quad 1 \leq i \leq n \tag{3.18}$$

be the canonical epimorphism. Observe that π has a section

$$\iota : \ \Bbbk \mathbb{G} \to \Bbbk G, \quad \prod_{i=1}^{n} \mathbb{g}_i^{i_j} \mapsto \prod_{i=1}^{n} g_i^{i_j} \tag{3.19}$$

which is not a group morphism. Let δ_L and $\triangleright$ be the comodule and module structure maps of $V \in {}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi}$. Define

$$\rho_L : \ V \to \Bbbk G \otimes V, \quad \rho_L = (\iota \otimes \mathrm{id})\delta_L$$
$$\blacktriangleright : \ \Bbbk G \otimes V \to V, \quad g \blacktriangleright Z = \pi(g) \triangleright Z$$

for all $g \in G$ and $Z \in V$. Then

Lemma 3.12. *$\widetilde{V} := (V, \rho_L, \blacktriangleright)$ is an object in ${}^{G}_{G}\mathcal{YD}^{\pi^*(\Phi)}$.*

Moreover we have

Proposition 3.13. *For any Nichols algebra $\mathcal{B}(V) \in {}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi}$, the Nichols algebra $\mathcal{B}(\widetilde{V}) \in {}^{G}_{G}\mathcal{YD}^{\pi^*(\Phi)}$ is isomorphic to $\mathcal{B}(V)$. Moreover, if $B(V)$ is of diagonal type, then $\mathcal{B}(\widetilde{V})$ is twist equivalent to a usual Nichols algebra.*

To summarize so far, we have found the route in Figure I of transforming a *nonassociative* Nichols algebra to a *usual* one:

$\mathcal{B}(V) \in {}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi}$ Original diagonal Nichols algebra

$\downarrow$

$\mathcal{B}(V) \cong \mathcal{B}(\widetilde{V}) \in {}^{G}_{G}\mathcal{YD}^{\pi^*(\Phi)}$ Lemma 3.11+Lemma 3.12

$\downarrow$ Proposition 2.5

$\mathcal{B}(\widetilde{V})$ is twisted equivalent to a usual Nichols algebra $\mathcal{B}(V)'$

Figure 1

According to this diagram, every diagonal Nichols algebra $\mathcal{B}(V) \in {}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi}$ corresponds to a usual diagonal Nichols algebra, denoted by $\mathcal{B}(V)'$ for convenience, in a canonical way.

Definition 3.14. The arithmetic root system of $\mathcal{B}(V)$ is defined to be that of $\mathcal{B}(V)'$. That is, $\triangle(\mathcal{B}(V))_{\chi,E} := \triangle(\mathcal{B}(V'))_{\chi,E}$ by the prescribed notations in Subsection 3.3. In particular, the root system $\triangle(\mathcal{B}(V))$ of $\mathcal{B}(V)$ equals to $\triangle(\mathcal{B}(V)')$.

In order to complete the classification of diagonal Nichols algebras, we need to find a return trip of the above diagram. To this end, we shall answer this question: For a usual diagonal type Nichols algebra $\mathcal{B}$ with arithmetic root system, when is $\mathcal{B}$ gotten from a Nichols algebra $\mathcal{B}(V) \in {}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi}$?

Firstly we observe the following fact.

LEMMA 3.15. [**28**, Lemma 4.9] *Let $\mathcal{B}(\widetilde{V}) \in {}^{G}_{G}\mathcal{YD}^{\pi^*(\Phi)}$ be a Nichols algebra of diagonal type and $\{Y_i|1 \leq i \leq m\}$ be a standard basis of $\widetilde{V}$. Then $\mathcal{B}(\widetilde{V})$ is isomorphic to a Nichols algebra in ${}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi}$ through π if and only if*

$$g_i^{\mathbb{m}_i} \blacktriangleright Y_j = Y_j, \quad 1 \leq i \leq n, 1 \leq j \leq m. \tag{3.20}$$

We point out that the result of Lemma 3.15 can also be deduced from [**8**, Proposition 4.3]. Now fix an usual Nichols algebra of diagonal type $\mathcal{B}(V)' \in {}^{G}_{G}\mathcal{YD}$ with support group G. According to Figure I, we need to answer the following question:

When is $\mathcal{B}(V)'^{J_{\underline{\mathbf{a}}}}$ isomorphic to a Nichols algebra in ${}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi_{\underline{\mathbf{a}}}}$ through π? (Question $(\diamond)$)

Let $\{X_i|1 \leq i \leq m\}$ be a standard basis of V. Assume that

$$\delta'_L(X_i) = h_i \otimes X_i, \quad g_k \triangleright' X_j = q_{kj} X_j$$

for $1 \leq i, j \leq m$, $1 \leq k \leq n$, $h_i \in G$ and $q_{kj} \in \mathbb{k}^*$, where δ'_L (resp. $\triangleright'$) is the comodule (resp. module) structure map of $\mathcal{B}(V)' \in {}^{G}_{G}\mathcal{YD}$. So there are $0 \leq x_{kj}, s_{ik} < m_k$ such that

$$q_{kj} = \zeta_{m_k}^{x_{kj}}, \quad h_i = \prod_{k=1}^{n} g_k^{s_{ik}}$$

for $1 \leq i, j \leq m$ and $1 \leq k \leq n$. Let $\mathbf{X} = (x_{ij})_{n \times m}$. By assumption, the support group $G_{\mathcal{B}(V)'} = G$, $\{h_i|1 \leq i \leq m\}$ generate the group G, so there are $t_{jl} \in \mathbb{N}$ such that

$$g_j = \prod_{l=1}^{m} h_l^{t_{jl}}, \quad 1 \leq j \leq n.$$

By $\mathbf{S}$ and $\mathbf{T}$, we denote the matrices $(s_{ik})_{m \times n}$ and $(t_{jl})_{n \times m}$. It is obvious that

$$\mathbf{TS} \equiv \begin{pmatrix} 1 \pmod{m_1} & 0 \pmod{m_1} & \cdots & 0 \pmod{m_1} \\ 0 \pmod{m_2} & 1 \pmod{m_2} & \cdots & 0 \pmod{m_2} \\ \cdots & \cdots & \cdots & \cdots \\ 0 \pmod{m_n} & 0 \pmod{m_n} & \cdots & 1 \pmod{m_n} \end{pmatrix}. \tag{3.21}$$

With these notations, we can give the answer to Question $(\diamond)$ now.

PROPOSITION 3.16. [**28**, Proposition 4.10] *The twisting $\mathcal{B}(V)'^{J_{\underline{\mathbf{a}}}}$ is isomorphic to a Nichols algebra in ${}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi_{\underline{\mathbf{a}}}}$ through π if and only if the following congruence equalities hold:*

$$\sum_{j=1}^{m} x_{ij} t_{lj} \equiv 0 \pmod{\mathbb{m}_i}, \ 1 \leq l < i \leq n, \tag{3.22}$$

$$\sum_{j=1}^{m} x_{ij} t_{ij} \equiv a_i \pmod{\mathbb{m}_i}, \ 1 \leq i \leq n, \tag{3.23}$$

$$(\sum_{j=1}^{m} x_{ij} t_{lj}) \mathbb{m}_l \equiv \mathbb{m}_i a_{il} \pmod{\mathbb{m}_i \mathbb{m}_l}, \ 1 \leq i < l \leq n. \tag{3.24}$$

PROPOSITION 3.17. [**28**, Corollary 4.12] *For the Nichols algebra $\mathcal{B}(V)' \in {}^{G}_{G}\mathcal{YD}$, there is at most one $\underline{\mathbf{a}} \in \mathcal{A}$ such that $\mathcal{B}(V)'^{J_{\underline{\mathbf{a}}}}$ is isomorphic to a Nichols algebra in*

${}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi_{\underline{\mathbf{a}}}}$ *through* π*. Moreover, this* $\underline{\mathbf{a}}$ *exists if and only if the Equations* (3.22) *hold and in this case* $\underline{\mathbf{a}}$ *can be taken in the following way:*

$$a_i \equiv \sum_{j=1}^{m} x_{ij}t_{ij} \ (\mathrm{mod}\ \mathbb{m}_i); \quad a_{il} \equiv \frac{\mathbb{m}_l}{\mathbb{m}_i}\sum_{j=1}^{m} x_{ij}t_{lj} \ (\mathrm{mod}\ \mathbb{m}_l); \quad a_{ilt} = 0 \tag{3.25}$$

for $1 \le i \le n$, $1 \le i < l \le n$ *and* $1 \le i < l < t \le n$.

Now we are in the position to find the "return trip" in Figure II.

$$\begin{tikzcd}
\mathcal{B}(V) \in {}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi_{\underline{\mathbf{a}}}} \\
\mathcal{B}(V)'^{J_{\underline{\mathbf{a}}}} \in {}^{G}_{G}\mathcal{YD}^{\pi^*(\Phi_{\underline{\mathbf{a}}})} \arrow[u, "\text{Eq. (3.22) satisfy}"'] \\
\text{Usual Nichols algebra } \mathcal{B}(V)' \in {}^{G}_{G}\mathcal{YD} \arrow[u, "\text{By Eq. (3.25), find } J_{\underline{\mathbf{a}}}"']
\end{tikzcd}$$

Figure II

Next we can give a complete classification of Nichols algebras of diagonal type with arithmetic root systems in ${}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi}$. Suppose $(\triangle, \chi, E)$ is an arithmetic root system, $\mathcal{D}_{\chi,E}$ is the Dynkin diagram of $(\triangle, \chi, E)$. Up to twist equivalence, $(\triangle, \chi, E)$ is uniquely determined by $\mathcal{D}_{\chi,E}$. Fix a Dynkin diagram with m vertices, we call

$$\{q_{ii} = \chi(e_i, e_i),\ \widetilde{q_{ij}} = \chi(e_i, e_j)\chi(e_j, e_i) | 1 \le i, j \le m\}$$

the structure constants of $\mathcal{D}_{\chi,E}$.

Definition 3.18. Let $\mathbb{G} = \mathbb{Z}_{\mathbb{m}_1} \times \cdots \times \mathbb{Z}_{\mathbb{m}_n}$ be the abelian group defined as above and set $m_i := \mathbb{m}_i^2$ for $1 \le i \le n$. Suppose $\mathcal{D}_{\chi,E}$ is a Dynkin diagram of an arithmetic root system $\triangle_{\chi,E}$ and $(q_{ii}, \widetilde{q_{ij}})$ is the set of structure constants. Assume that there exist parameter matrices $\mathbf{S}$ and $\mathbf{X}$ satisfying

1. $\mathbf{S} = (s_{ij})_{m\times n}$ is a matrix with integer entries $0 \le s_{ij} < m_i$ for all $1 \le i \le m, 1 \le j \le n$ such that there exists a matrix $\mathbf{T} = (t_{ij})_{n\times m}$ satisfying (3.21).
2. $\mathbf{X} = (x_{ij})_{n\times m}$ with integer entries $0 \le x_{ij} < m_i$ for all $1 \le i \le m, 1 \le j \le n$ such that $q_{ii} = \prod_{k=1}^{n} \zeta_{m_k}^{s_{ik}x_{ki}}$, $\widetilde{q_{ij}} = \prod_{k=1}^{n} \zeta_{m_k}^{s_{ik}x_{kj}+s_{jk}x_{ki}}$, and satisfy Equations (3.22).

Then we call $\mathfrak{D} = \mathfrak{D}(\mathcal{D}_{\chi,E}, \mathbf{S}, \mathbf{X})$ a root datum over $\mathbb{G}$, and $\triangle_{\chi,E}$ the arithmetic root system of $\mathfrak{D}$.

For a fixed root datum $\mathfrak{D} = \mathfrak{D}(\mathcal{D}_{\chi,E}, \mathbf{S}, \mathbf{X})$ over $\mathbb{G}$, define a sequence $\underline{\mathbf{a}} \in A$ through Equations (3.25). Now we can define a Nichols algebra $\mathcal{B}(V_{\mathfrak{D}}) \in {}^{G}_{G}\mathcal{YD}^{\pi^*(\Phi_{\underline{\mathbf{a}}})}$ in the following way: Let $V_{\mathfrak{D}}$ be the Yetter-Drinfeld module in ${}^{G}_{G}\mathcal{YD}^{\pi^*(\Phi_{\underline{\mathbf{a}}})}$ with a standard basis $\{X_i | 1 \le i \le m\}$ such that

$$\delta_L(X_i) = \prod_{k=1}^{n} g_k^{s_{ik}} \otimes X_i, \quad g_i \blacktriangleright X_j = \zeta_{m_i}^{x_{ij}} \frac{J_{\underline{\mathbf{a}}}(g_i, \prod_{k=1}^{n} g_k^{s_{ik}})}{J_{\underline{\mathbf{a}}}(\prod_{k=1}^{n} g_k^{s_{ik}}, g_i)} X_j.$$

Theorem 3.19. [**28**, Theorem 4.14]

(1) *The Nichols algebra $\mathcal{B}(V_{\mathfrak{D}})$ is isomorphic to a Nichols algebra of diagonal type with arithmetic root system in ${}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi_{\underline{\mathbf{a}}}}$ through the group epimorphism $\pi : G \to \mathbb{G}$.*

(2) *Suppose $\mathcal{B}(V)$ is a Nichols algebra of diagonal type with arithmetic root system in ${}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi_{\underline{\mathbf{a}}}}$ and the support group is $\mathbb{G}$, then there exists a root datum $\mathfrak{D}$ over $\mathbb{G}$ such that $\mathcal{B}(V_{\mathfrak{D}}) \cong \mathcal{B}(V)$ through the group epimorphism $\pi : G \to \mathbb{G}$.*

By this theorem, we know that the Nichols algebra $\mathcal{B}(V_{\mathfrak{D}})$ is isomorphic to a unique Nichols algebra in ${}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi_{\underline{\mathbf{a}}}}$. For convenience, this Nichols algebra is denoted by $\mathcal{B}(\mathfrak{D})$ and $\mathbb{M}(\mathfrak{D}) = \mathcal{B}(\mathfrak{D})\#\Bbbk\mathbb{G}$.

With the previous preparation, now we are in the position to give the classification result of connected finite-dimensional pointed coquasi-Hopf algebras of diagonal type. Suppose that $\mathbb{M}$ is a finite-dimensional connected pointed coquasi-Hopf algebras of diagonal type with associator $\Phi_{\underline{\mathbf{a}}}$, $\mathbb{G} = G(\mathbb{M})$ and $\mathcal{R}$ the coinvariant subalgebra of $\mathbb{M}$. Note that $\mathcal{R} = \sum_{i\geq 0} \mathcal{R}_i$ is a graded Hopf algebras in ${}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi_{\underline{\mathbf{a}}}}$. In fact we have

Proposition 3.20. [**28**, Lemma 4.1, Proposition 5.1] *$\Phi_{\underline{\mathbf{a}}}$ is an abelian 3-cocycle on $\mathbb{G} = G(\mathbb{M})$, and $\mathcal{R} \cong \mathcal{B}(\mathcal{R}_1)$ is a Nichols algebra in ${}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi_{\underline{\mathbf{a}}}}$.*

Combining Theorem 3.19 and Proposition 3.20, we obtain our main classification result.

Theorem 3.21. [**28**, Theorem 5.7] *Keep the notations as before. We have*

(1). *The coquasi-Hopf algebra $\mathbb{M}(\mathfrak{D})$ is a connected coradical graded pointed coquasi-Hopf algebra of diagonal type over the group $\mathbb{G}$. Moreover, $\mathbb{M}(\mathfrak{D})$ is finite-dimensional if and only if the heights of all restricted Poincare-Birkhoff-Witt generators of $\mathcal{B}(\mathfrak{D})$ are finite.*

(2). *Any finite-dimensional connected coradical graded pointed coquasi-Hopf algebra of diagonal type over $\mathbb{G}$ is isomorphic to a $\mathbb{M}(\mathfrak{D})$ for some $\mathfrak{D}$.*

In [**3**, Conjecture 1.4], Andruskiewitch-Schneider conjectured that every finite-dimensional pointed Hopf algebras over $\Bbbk$ is generated by group-like and skew-primitive elements. This is the so called generation in degree one problem, which plays an important role on the classification of pointed Hopf algebras. It is true in many cases, see [**6**]. This conjecture was generalized to finite-dimensional pointed coquasi-Hopf algebras or even to pointed finite tensor categories [**16**].

Corollary 3.22. [**28**, Corollary 5.8] *Suppose $\mathbb{M}$ is a finite-dimensional pointed coquasi-Hopf algebras of diagonal type, then $\mathbb{M}$ is generated by group-like and skew-primitive elements.*

To conclude this section, we remark that there are many new classes of genuine finite-dimensional coradically graded pointed coquasi-Hopf algebras constructed in [**27**, **28**] explicitly.

4. Finite quasi-quantum groups of nondiagonal type: A first attempt

In this section, we will present some classification results of nondiagonal finite quasi-quantum groups over abelian groups obtained in [**33**]. This is an initial step to go further beyond [**27**, **28**].

In the rest of this section, let G be a finite abelian group and Φ be a nonabelian 3-cocycle on G. Similar to the diagonal case, the crux is a complete understanding of the Nichols algebras of all $V \in {}^G_G\mathcal{YD}^\Phi$. To this end, in principle we need to develop a theory for the Nichols algebras of semisimple twisted Yetter-Drinfeld modules. The Hopf version of such a theory was developed in [**4**, **23**]. However, at present it seems not easy to extend this theory to the quasi-Hopf case directly. As a trial step, firstly we study the Nichols algebras of semisimple twisted Yetter-Drinfeld modules with few summands. It turns out that if the number of summands is less than or equal to 2, then we are able to make a connection from this to the diagonal case. The main idea is to consider the support groups of such easy Yetter-Drinfeld modules and carry out the base group change as in our previous works [**27**, **28**]. More precisely, if $V \in {}^G_G\mathcal{YD}^\Phi$ is nondiagonal and has at most 2 simple summands, then its support group G_V is either a cyclic group or the direct product of two cyclic groups. Moreover, the Nichols algebra $\mathcal{B}(V) \in {}^G_G\mathcal{YD}^\Phi$ is essentially nothing other than $\mathcal{B}(V) \in {}^{G_V}_{G_V}\mathcal{YD}^{\Phi|_{G_V}}$. In this situation, all 3-cocycles on G_V are abelian and then [**27**, **28**] can be applied.

Our first main result is a complete clarification of the Nichols algebra $\mathcal{B}(V)$ when V is a simple twisted Yetter-Drinfeld module of nondiagonal type. In particular, we provide an explicit necessary and sufficient condition on V for $\mathcal{B}(V)$ to be finite-dimensional. The same idea and process can be applied to $\mathcal{B}(V)$ when V is a direct sum of 2 simple twisted Yetter-Drinfeld modules. As this will not provide more insights for our ultimate aim, we do not include a detailed discussion of this case. Instead, we present several simple examples to offer the reader some flavor. Surprisingly, the result on $\mathcal{B}(V)$ with V simple is already enough for us to achieve half of our final aim. Our second main result is a complete classification of finite-dimensional coradically graded pointed coquasi-Hopf algebras over abelian groups of odd order. The key observation is that $\mathcal{B}(V) \in {}^G_G\mathcal{YD}^\Phi$ is infinite-dimensional for any simple nondiagonal twisted Yetter-Drinfeld module V if the order of G is odd. As an application, we also prove that any pointed finite tensor category over an abelian group of odd order is tensor generated by objects of length 2, which partially confirms the generation conjecture [**16**, Conjecture 5.11.10.] of pointed finite tensor categories due to Etingof, Gelaki, Nikshych and Ostrik.

4.1. The Nichols algebras of simple twisted Yetter-Drinfeld modules. In this subsection we focus on the Nichols algebras of nondiagonal Yetter-Drinfeld modules. Note that if Φ is an abelian 3-cocycle on G, then each object of ${}^G_G\mathcal{YD}^\Phi$ is of diagonal type. So nondiagonal Yetter-Drinfeld modules appear in ${}^G_G\mathcal{YD}^\Phi$ only if Φ is nonabelian. The following proposition is an immediate consequence of Propositions 2.4 and (2.3).

PROPOSITION 4.1. *Suppose that G is a cyclic group $\mathbb{Z}_m$ or a direct product of two cyclic groups, say $\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2}$, then all the 3-cocycles on G are abelian.*

PROPOSITION 4.2. [**33**, Proposition 3.16] *Let G be a finite abelian group, Φ a 3-cocycle on G. Suppose that $\mathcal{B}(V)$ is a Nichols algebra in ${}^G_G\mathcal{YD}^\Phi$, where V is a simple Yetter-Drinfeld module, or a direct sum of two simple Yetter-Drinfeld modules. Then $\mathcal{B}(V)$ is isomorphic to a Nichols algebra of diagonal type $\mathcal{B}(V')$ in ${}^H_H\mathcal{YD}^\Psi$, where $H = G_V$ and $\Psi = \Phi|_H$.*

According to this proposition, we can apply the theory of Nichols algebras of diagonal type to study the Nichols algebras of simple twisted Yetter-Dinfeld modules, or of a direct sum of two simple twisted Yetter-Drinfeld modules. If $V = V_g$ is a simple object in ${}^G_G\mathcal{YD}^\Phi$, then we denote $g_V = g$.

PROPOSITION 4.3. [**33**, Proposition 3.18] *Let G be a finite abelian group, Φ a 3-cocycle on G. Suppose V is a simple Yetter-Drinfeld module of nondiagonal type in ${}^G_G\mathcal{YD}^\Phi$, $g_V = g$. Then $\mathcal{B}(V)$ is finite-dimensional if and only if V is one of the following two cases:*

(C1) *$g \triangleright v = -v$ for all $v \in V$;*
(C2) *$\dim(V) = 2$ and $g \triangleright v = \zeta_3 v$ for all $v \in V$, here ζ_3 is a 3-rd primitive root of unity.*

There exist Yetter-Drinfeld modules satisfying conditions C1 or C2, see [**33**, Example 3.19-3.20].

4.2. Finite quasi-quantum groups over abelian groups of odd order. In this subsection we provide a complete classification of finite-dimensional coradically graded pointed coquasi-Hopf algebras over abelian groups of odd order. This is also applied to the classification theory of pointed finite tensor categories. In particular, we give a partial answer to the following

CONJECTURE 4.4. [**16**, Conjecture 5.11.10.] *A pointed finite tensor category is tensor generated by objects of length* 2.

This conjecture is due to Etingof, Gelaki, Nikshych and Ostrik, hence will be called EGNO's conjecture in the following. It is a natural generalization of the well known Andruskiewitsch-Schneider conjecture [**3**, Conjecture 1.4] due to the following proposition.

PROPOSITION 4.5. [**33**, Proposition 4.10] *Suppose that M is a finite-dimensional pointed coquasi-Hopf algebra. Then M is generated by group-like and skew-primitive elements if and only if* $\mathrm{comod}(M)$ *is tensor generated by objects of length* 2.

Now we need to get more information of Nichols algebras in ${}^G_G\mathcal{YD}^\Phi$. Applying linear representation theory of G, we prove the following important proposition.

PROPOSITION 4.6. *For each simple object V in ${}^G_G\mathcal{YD}^\Phi$, we have* $\dim(V) \mid |G|$.

Note that if $|G|$ is odd, the order of an element in G is also odd, hence there is not a simple object satisfying condition C1 of Proposition 4.3. According to Proposition 4.6, $\dim(V) \mid |G|$ implies $\dim(V)$ is also odd, and hence V doesn't satisfy condition C2 of Proposition 4.3. So we have the following proposition.

PROPOSITION 4.7. *Let G be a finite abelian group of odd order and Φ be a 3-cocycle on G. Suppose that $V \in {}^G_G\mathcal{YD}^\Phi$ is not diagonal. Then $\mathcal{B}(V)$ is infinite-dimensional.*

This proposition implies that if $|G|$ is odd, then every finite-dimensional Nichols algebra in ${}^G_G\mathcal{YD}^\Phi$ must be of diagonal type. So according to Proposition 3.20, we can easily prove the following theorem.

THEOREM 4.8. [**33**, Theorem 4.11] *Suppose that $R = \oplus_{i\geq 0} R[i]$ is a finite-dimensional connected coradically graded braided Hopf algebra in ${}^G_G\mathcal{YD}^\Phi$, where G is an abelian group of odd order and Φ is a 3-cocycle on G. Then $R = \mathcal{B}(R[1])$.*

Combining Proposition 4.5 and Theorem 4.8, we obtain a partial answer to Conjecture 4.4.

THEOREM 4.9. [**33**, Theorem 4.2] *Suppose that $\mathcal{C}$ is a pointed finite tensor category with $G(\mathcal{C})$ an abelian group of odd order. Then $\mathcal{C}$ is tensor generated by objects of length* 2.

With the help of Theorem 4.8, we achieve the classification of coradically graded pointed coquasi-Hopf algebras and that of coradically graded pointed finite tensor categories over abelian groups of odd order. Here a tensor category $\mathcal{C}$ is said to be coradically graded if $\mathcal{C}$ is equivalent to the category of comodules over a coradically graded coquasi-Hopf algebra.

Let $\Delta_{\chi,E}$ be an arithmetic root system. For each positive root $\alpha \in \Delta$, define $q_\alpha = \chi(\alpha, \alpha)$. Then the height of α is defined by

$$\text{ht}(\alpha) = \begin{cases} |q_\alpha|, & \text{if } q_\alpha \neq 1 \text{ is a root of unity;} \\ \infty, & \text{otherwise.} \end{cases} \tag{4.1}$$

A function $\chi : G \longrightarrow \Bbbk^*$ is called a **quasi-character** associated to a 2-cocycle ω on G if for all $f, g \in G$,

$$\chi(f)\chi(g) = \omega(f, g)\chi(fg), \quad \chi(1) = 1. \tag{4.2}$$

It is clear that there is a quasi-character associated to ω if and only if ω is symmetric. Recall that for a fixed 3-cocycle Φ on G, $\{\widetilde{\Phi}_g | g \in G\}$ gives 2-cocycles on G.

DEFINITION 4.10. Let $\chi_1, \cdots, \chi_n$ be quasi-characters of G associated to $\widetilde{\Phi}_{g_1}, \cdots, \widetilde{\Phi}_{g_n}$ respectively. We say the series $(\chi_1, \cdots, \chi_n)$ is of finite type if there is an arithmetic root system $\Delta_{\chi,E}$ of rank n such that:

- $\chi_i(g_j)\chi_j(g_i) = q_{ij}q_{ji}$, $\chi_i(g_i) = q_{ii}$ for all $1 \leq i, j \leq n$. Here $q_{ij} = \chi(e_i, e_j)$ for $e_i, e_j \in E$.
- $\text{ht}(\alpha) < \infty$ for all $\alpha \in \Delta$.

For a series of quasi-characters $(\chi_1, \cdots, \chi_n)$ of finite type associated to $\widetilde{\Phi}_{g_1}, \cdots, \widetilde{\Phi}_{g_n}$, we can attach to it a twisted Yetter-Drinfeld module $V(\chi_1, \cdots, \chi_n)$ with a standard basis $\{X_1, \cdots, X_n\}$ such that $g_i \triangleright X_j = \chi_j(g_i)X_j$ and $\delta_L(X_i) = g_i \otimes X_i$ for all $1 \leq i, j \leq n$. Now we can give the classification result.

THEOREM 4.11. [**33**, Theorem 4.13] *Let G be a finite abelian group of odd order, Φ a 3-cocycle on G.*

(1) *If $(\chi_1, \cdots, \chi_n)$ is a series of quasi-characters of finite type associated to the 2-cocycles $\widetilde{\Phi}_{g_1}, \cdots, \widetilde{\Phi}_{g_n}$, then $\mathcal{B}(V(\chi_1, \cdots, \chi_n))$ is a finite-dimensional Nichols algebra in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}^\Phi$.*

(2) *Suppose that $\mathcal{C}$ is a coradically graded pointed finite tensor category such that $G(\mathcal{C}) = G$ and the associator is Φ. Then there exists a series of quasi-characters $(\chi_1, \cdots, \chi_n)$ of finite type associated to $\widetilde{\Phi}_{g_1}, \cdots, \widetilde{\Phi}_{g_n}$ such that*

$$\mathcal{C} \cong \text{comod}(\mathcal{B}(V(\chi_1, \cdots, \chi_n))\#\Bbbk G).$$

5. Some further problems

Finally, we propose some further research problems which are natural extensions of our previous works.

Problem 1. Pursuit a complete classification of finite-dimensional coradically graded pointed coquasi-Hopf algebras of nondiagonal type over abelian groups. Thanks to [**33**], it remains to consider those over even order abelian groups. Moreover, according to some examples we worked out, it is quite possible that the Nichols algebra of a twisted Yetter-Drinfeld module with at least 3 nondiagonal simple summands is infinite-dimensional. This implies that nontrivial finite quasi-quantum groups of nondiagonal type over abelian groups are very rare. Thus, a complete classification seems possible.

Problem 2. Consider pointed finite tensor categories and finite quasi-quantum groups over nonabelian groups. To the best of our knowledge, there seems not even a single example of finite-dimensional connected pointed coquasi-Hopf algebra with nontrivial associator over nonabelian groups in the literature. In addition, the cohomology of finite nonabelian groups seems much more complicated. For some easy classes of nonabelian groups, e.g. the semi-direct product of two cyclic groups, the idea of Section 2 may be applied to obtain unified and explicit 3-cocycle formulas. Finite quasi-quantum groups over such groups may be workable. In pursuit of a general theory as the abelian case, it seems worthwhile to develop the general theory of Nichols algebras of semisimple twisted Yetter-Drinfeld modules, as well as the related Weyl groupoids and arithmetic root systems.

Problem 3. Consider the lifting or deformation theory of coradically graded finite quasi-quantum groups. As far as we know, the lifting of (co)quasi-Hopf algebras was considered only in [**5**, **15**]. There are not many results in this direction. The problem sees very complicated and challenging. An obvious difficulty lies in associators for which there is no suitable tools to control yet. One may try to apply the deformation and related cohomology theory of quasi-quantum groups. We also wonder if there is a nice theory of cocycle deformations of quasi-quantum groups.

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